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Galilean Duffin–Kemmer–Petiau algebra and symplectic structure

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Abstract
We develop the Duffin–Kemmer–Petiau (DKP) approach in the phase-space picture of quantum mechanics by considering DKP algebras in a Galilean covariant context. Specifically, we develop an algebraic calculus based on a tensor algebra defined on a five-dimensional space which plays the role of spacetime background of the non-relativistic DKP equation. The Liouville operator is determined and the Liouville–von Neumann equation is written in two situations: the free particle and a particle in an external electromagnetic field. A comparison between the non-relativistic and the relativistic cases is commented.

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1. Introduction

Over the last decade we have noted a renewed interest in the Duffin–Kemmer–Petiau (DKP) equation and its corresponding algebra [1–10]. Among the reasons for such an interest are: (i) the practical importance of the DKP equation as a part of a program to study the interaction of spin S = 0, 1 hadrons with nuclei [1, 3, 8, 11, 12]; (ii) the application of the DKP approach to the quantum chromodynamics (QCD) (large and short distances) [6]; (iii) the study of the DKP algebras from a modern perspective [2] and as a way of understanding the nature of the relativistic phase space [13] which is of interest for the formulation of the hadronic quantum theory [14]; and (iv) the study of the representations of the DKP algebras in order to determine a corresponding DKP first-order wave equation in the non-relativistic context [5, 15]. Furthermore, the study of the DKP algebras has provided a way of describing relativistic quantum systems via Liouville–von Neumann equations written for amplitudes over the phase-space manifold [4, 13]. This in turn leads to the Wigner-function formalism. In this context one notes the important role played by the so-called geometric (Clifford and Grassmann) algebras as a way to describe and characterize symplectic structures which

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allow the description of elementary physical systems carrying internal degrees of freedom [13, 16, 17, 18]. In addition, the use of geometric algebras in connection with the Galilean DKP approach may provide an alternative prescription to analyse the (not completely solved) problem of the association among representations of DKP-algebras and spin.

In the algebraic phase-space picture of spinning-particle equations, the spinorial character of the theory is contained essentially in the algebraic structure of the Liouville operator in the sense that the spin algebras are combined with the differential operators involved in the Liouville equation. This combined structure is then what we shall mean by a Liouville (or Liouville–von Neumann)-like equation in a generalized phase space. Such space appears as a product of the usual space of (q, p) variables with a space describing the internal degrees of freedom. The Liouville operator then operates on functions over that extended phase space. These functions are constructed here using the Wigner–Moyal transformation. Such a program has been suggested by Bohm and Hiley [16, 19] and Holland [17] and applied by Holland [18], Fernandes and Vianna [13] to derive Louville-like equations in the generalized relativistic phase space. Holland takes the complex Dirac algebra C_4 and obtains the phase-space picture of the Dirac and Feynman–Gell–Mann equations for massive spin $1/2$ particles. Fernandes and Vianna considered the DKP algebra D_r^{r+1} and obtained the phase-space picture of the DKP particles with internal degrees of freedom. However, such developments have not been carried out in the non-relativistic realm yet. The difficulties in the case rely on the non-geometric characterization of the non-relativistic physics as it is usually formulated. This kind of problem has been addressed here by using the notion of Galilean covariance, which has been proposed to equip the Galilean spacetime with a geometrical (and so a tensor) structure [20–23]. In particular, Santana *et al* [23] introduced the notion of Galilean covariance through an immersion of the Euclidean space, $\mathbf{R}^3$, in a $(4+1)$-de Sitter space, say G, characterized by a metric given by

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{1}$$

Therefore, the Galilei symmetries emerge as a subgroup of the non-homogeneous linear transformations in G. An interesting aspect in this formalism is that the Galilean covariance is manifested throughout and quite similar to the relativistic formulation. As a consequence non-relativistic first-order wave equations can be derived [15] following the Bhabha approach [24, 25] (which was first introduced in the relativistic context). In particular, a non-relativistic DKP equation and the corresponding algebra can be obtained [5]. Using this Galilean covariant formulation we show in the present paper that it is possible to obtain the phase-space representation of the DKP equation. In doing so we obtain the following results: (i) the Liouville operator associated with the DKP equation; (ii) the effect of the electromagnetic field on the internal degrees of freedom; and (iii) the classification of the dimensions for representations of the non-relativistic DKP algebras. To our knowledge these results are new in the non-relativistic context and may contribute to obtain a clear-cut distinction between the specific relativistic features of the theory [13, 16, 18] and those which equally follow from a consistent non-relativistic treatment.

The presentation of the paper is as follows. In section 2, we set forth the notation and discuss some basic concepts about G and consider T^*G equipped with a symplectic structure. In section 3, using geometric algebras, we consider the non-relativistic DKP algebra such that the elements of the representation space are defined from the minimal ideal of an extended

Grassmann algebra. This construction is reminiscent of the discussion of DKP equation using the language of the Clifford algebra of inhomogeneous exterior forms, which is related to the geometric algebra [26]. In section 4, we derive the non-relativistic DKP equation in the phase space. These results are used with the notion of Wigner–Moyal transformation to perform the derivation of a phase-space Liouville–von Neumann equation in section 5. The connection with a gauge field is considered in section 6. Final remarks and conclusions are presented in section 7.

2. Galilean covariance: outline and notation

In the Galilean covariant approach [27], a 5-momentum is specified by $p_\mu = (\mathbf{p}, p_4, p_5)$, where $\mathbf{p}$ stands for the Euclidean vector momentum, $-p_4 = H$ the energy and $-P_5 = m$ the mass. Using the metric given in equation (1), we have the dispersion relation $p_\mu p^\mu = p_\mu p_\nu g^{\mu\nu} = \mathbf{p}^2 - 2p^4 p^5 = k^2$, where k is a constant. This relation satisfies the physical requirement that for a free non-relativistic particle we have $H = \mathbf{p}^2/2m$; in other words $p_\mu p^\mu = \mathbf{p}^2 - 2mH = 0$. As a consequence, k can be made zero, or absorbed in H.

A canonical coordinate associated with p_μ is given by a 5-vector in G written as $q^\mu = (\mathbf{q}, q^4, q^5)$. The entries in q^μ are then physically interpreted as follows: $\mathbf{q}$ stands for the canonical coordinate attached to $\mathbf{p}$; q^4 is the canonical coordinate associated with H, and so q^4 can be considered as the time coordinate; q^5 is the canonical coordinate associated with m, and is explicitly given in terms of $\mathbf{q}$ and q^4 according to the following reasoning. Using the canonical coordinates, the distance in G is $q_\mu q^\mu = q^\mu q^\nu g_{\mu\nu} = \mathbf{q}^2 - 2q^4 q^5 = s^2$. Then, in order to take $s = 0$ (the counterpart of the dispersion relation written in terms of q^μ), we have $q^5 = \mathbf{q}^2/2q^4$. Considering $q^4 = vt$, where v is a velocity used to fix units, it follows that $q^5 = \mathbf{q}^2/2vt$.

The content of canonical coordinate can be made more precise here using the notion of symplectic structure [28]. Considering in G the set of 5-vector q^μ, we define in the cotangent bundle T^*G a symplectic structure via the 2-form ω,

$$\omega = \mathrm{d}q^\mu \wedge \mathrm{d}p_\mu = g^{\mu\nu}\,\mathrm{d}q_\nu \wedge \mathrm{d}p_\mu \qquad \mu, \nu = 1, 2, \ldots, 5. \tag{2}$$

The symplectic structure ω on T^*G provides a one-to-one correspondence, $X \to \omega(X)$ (the symplectic metric contraction) between vector fields X and linear differential forms $\omega(X)$. The distinguished class of *Hamiltonian* vector fields is the class of vector fields X on T^*G that corresponds to one-forms derived from functions, provided that T^*G has vanishing first cohomology group [29]. But at least *locally* X is of the form

$$X_f = \frac{\partial f}{\partial q^\mu}\frac{\partial}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu}\frac{\partial}{\partial q^\mu}$$

where f is a C^∞ function in the ten-dimensional (phase space) manifold T^*G with coordinates (q^μ, p_μ). We have

$$\begin{aligned}\omega(X_f, X_h) &= \{f, h\}\\ &= \mathrm{d}q^\nu(X_f)\,\mathrm{d}p_\nu(X_h) - \mathrm{d}p_\nu(X_f)\,\mathrm{d}q^\nu(X_h)\\ &= g^{\mu\nu}\left(\frac{\partial f}{\partial q^\mu}\frac{\partial h}{\partial p^\nu} - \frac{\partial h}{\partial q^\mu}\frac{\partial f}{\partial p^\nu}\right)\end{aligned}$$

where $\{f, h\}$ is the Poisson bracket of the functions f and h on T^*G. Observe that, since $\omega(X_h) = \mathrm{d}h$, we have $\{f, h\} = \mathrm{d}f(X_h) = X_h f$.

Classical mechanics on T^*G is derived in the Liouville representation by the flow

$$\partial_\mu f := \frac{\partial f}{\partial q^\mu} = -X_{p_\mu} f \tag{3}$$

where $f(q,p)$ is a real (C^∞) density distribution function on T^*G, and the (five-dimensional) vectors q and p are defined (here, we assume $v=1$ for convenience) by $q^\mu=(\mathbf{q},t,\mathbf{q}^2/2t)$ and $p^\mu=(\mathbf{p},m,H)$, where $H=\mathbf{p}^2/2m$. Then, in terms of components, we have from equation (3),

$$\partial_i f=-X_{p_i}f\mapsto\partial_i f=\{p_i,f\}\tag{4}$$

$$\partial_4 f=-X_{p_4}f\mapsto\partial_t f=\{H,f\}\tag{5}$$

$$\partial_5 f=-X_{p_5}f\mapsto\partial_5 f=0.\tag{6}$$

Consistency relations are given by equations (4) and (6), whereas equation (5) is nothing but the Liouville equation.

The set of linear non-homogeneous transformations in G admits 15 generators fulfilling an algebra isomorphic to the non-homogeneous $so(4+1)$. This algebra has the extended Galilei algebra in (3 + 1) dimensions as a subalgebra [27, 28]. In the next section we use G as the basis to build geometric algebras for non-relativistic theories.

3. Geometric Galilean DKP algebras

In this section we analyse the emergence of DKP algebras having a (4 + 1)-de Sitter space G as the underlying manifold of geometric algebras. We present initially a general construction [13] and finally we specify our relations to G. Let us consider an n-dimensional vector space V and its dual, say $V^*=U$. Let W be the vector space, $W=V\oplus U$. From W, a canonical symmetric bilinear form, $B:W\times W\to\mathbf{R}$ (or $\mathbf{C}$), is defined by

$$B[(w),(w')]=B[(v,u),(v',u')]=\tfrac{1}{2}(\langle u,v'\rangle+\langle u',v\rangle)\tag{7}$$

where $v,v'\in V$, $u,u'\in U$, $w=(v,u)$, $w'=(v',u')\in W$ and $\langle\cdot,\cdot\rangle$ is the natural pairing of vectors and covectors. The choice of a bilinear form on a vector space equips this vector space with a Clifford algebra structure. Succinctly, one first includes W into a tensor algebra

$$T(W)=\bigoplus_{i=0}^{\infty}T^i(W)$$

by identifying W with the one piece $T^1(W)$ of $T(W)$. The Clifford algebra $C(W)$ is then defined as the quotient algebra $C(W)=T(W)/I$, where I is the two-side ideal,

$$I=w\otimes w'+w'\otimes w-2B(w,w').$$

As a consequence, we have

$$[w,w']_+=2B(w,w')\tag{8}$$

where $[\cdot,\cdot]_+$ is the anticommutator. Taking into account equation (7), equation (8) is rewritten as

$$[v,u]_+=\langle v,u\rangle\tag{9}$$

$$[u,u']_+=[v,v']_+=0.\tag{10}$$

This result has the effect of exhibiting the algebra in terms of vectors and covectors.

Let us then consider $(e_1,\ldots,e_n)$ a basis for V, and $(e^1,\ldots,e^n)$ a basis for $U(=V^*)$. Now, we can write the product rules (9) and (10) for the basis elements getting

$$[e_i,e^j]_+\equiv e_ie^j+e^je_i=\delta_i^j1$$
$$[e_i,e_j]_+\equiv e_ie_j+e_je_i=0$$
$$[e^i,e^j]_+\equiv e^ie^j+e^je^i=0.$$

We then consider the primitive idempotent

$$P = N_1 \cdots N_n$$

where $N_i = e_i e^i$ (no sum on the indices). A canonical basis for $C(W)$ can be given by the 2^{2n} elements

$$P^{k_1\ldots k_s}_{j_1\ldots j_r} = e^{k_1} \cdots e^{k_s} P e_{j_r} \cdots e_{j_1}$$

which satisfy

$$P^{k_1\ldots k_s}_{j_1\ldots j_r} P^{i_1\ldots i_s}_{h_1\ldots h_r} = \delta_{r,s} \delta^{i_1\ldots i_s}_{j_1\ldots j_r} P^{k_1\ldots k_s}_{h_1\ldots h_r}.$$

A basis for the minimal left ideal of $C(W)$ results from a projection of the basis $P^{k_1\ldots k_s}_{j_1\ldots j_r}$ by the primitive idempotent P, which gives rise to elements of form $\Gamma P = \Psi$ for every $\Gamma \in C(W)$. In terms of components we have for Γ,

$$\Gamma = \sum_{r,s=0}^{n} \frac{1}{r!s!} A^{j_1\ldots j_r}_{k_1\ldots k_s} P^{k_1\ldots k_s}_{j_1\ldots j_r}$$

such that

$$\Gamma P = \Psi = \sum_{r,s=0}^{n} \frac{1}{r!s!} A^{j_1\ldots j_r}_{k_1\ldots k_s} P^{k_1\ldots k_s}_{j_1\ldots j_r}(P) = \sum_{s=0}^{n} \frac{1}{s!} A_{k_1\ldots k_s} P^{k_1\ldots k_s} \tag{11}$$

where $P^{k_1\ldots k_s} = e^{k_1} \cdots e^{k_s} P$. The space $C_P(W)$ of elements Ψ plays the role of a space of representation for $C(W)$. Indeed, it can be easily verified since that $\Gamma : C_P(W) \to C_P(W)$, such that $\Psi' = \Gamma\Psi$, with $\Psi, \Psi' \in C_P(W)$ and $\Gamma \in C(W)$.

Now we can construct the DKP algebras from the basis $P^{k_1\ldots k_s}_{j_1\ldots j_r}$. The elements

$$\Pi_r = r! P^{j_1\ldots j_r}_{j_1\ldots j_r} \tag{12}$$

are idempotent elements in $C(W)$. They expand the unity of the algebra, i.e.

$$1 = \sum_{r=0}^{n} \Pi_r \qquad \Pi_0 = P.$$

These idempotents Π_r satisfy the following properties:

$$\Pi_r \Pi_s = \delta_{rs} \Pi_s \tag{13}$$

$$e^i \Pi_r = \Pi_{r+1} e^i \tag{14}$$

$$\Pi_r e_i = e_i \Pi_{r+1}. \tag{15}$$

At this point we specialize our presentation to derive the main results of this section. Consider $V \equiv G$, where $n = \dim G = 5$, such that $r = 0, 1, \ldots, 5$ and $W = G \oplus G^*$. With the projectors given in equation (12) and the metric introduced in equation (1), it is possible to obtain the DKP algebra by defining

$$\beta^{(r)}_\mu = \Pi_r e_\mu + g_{\nu\mu} e^\nu \Pi_r = e_\mu \Pi_{r+1} + g_{\nu\mu} e^\nu \Pi_r. \tag{16}$$

With this definition for the β's and using the properties given in equations (13)–(15), we can show that

$$\beta^{(r)}_\mu \beta^{(r)}_\nu \beta^{(r)}_\rho + \beta^{(r)}_\rho \beta^{(r)}_\nu \beta^{(r)}_\mu = g_{\mu\nu} \beta^{(r)}_\rho + g_{\rho\nu} \beta^{(r)}_\mu \tag{17}$$

which is the fundamental relation of the DKP algebra, but with $g_{\mu\nu}$ given in equation (1). The DKP algebra is then generated by the $\beta^{(r)}_\mu$ and $\Pi_r + \Pi_{r+1}$ which represent the unity of the algebra. The algebra is irreducible when $r \neq (n-1)/2$, because it is a total matrix algebra of a direct sum of spaces of dimension r and $r+1$ which we denote as D_r^{r+1}. It is an $m \times m$ matrix algebra with

$$m = \left[\binom{n}{r} + \binom{n}{r+1}\right] \tag{18}$$

(see [13, 30] for further details). Each value of r fixes a representation, describing in principle different particles. For $r = 0$ we have a scalar particle and $r = 1$ the vector one, for instance.

With $\beta_\mu^{(r)}$ defined above in equation (16), let us introduce

$$\eta_i^{(r)} = 2\big(\beta_i^{(r)}\big)^2 - 1 \qquad i = 1, 2, 3 \tag{19}$$

$$\eta_4^{(r)} = \big(\beta_4^{(r)} - \beta_5^{(r)}\big)^2 - 1 \tag{20}$$

$$\eta_5^{(r)} = \big(\beta_4^{(r)} + \beta_5^{(r)}\big)^2 + 1. \tag{21}$$

We can hence show these η's satisfy the following properties:

$$\beta_i^{(r)}\eta_i^{(r)} = -\eta_i^{(r)}\beta_i^{(r)} \tag{22}$$

$$\beta_4^{(r)}\eta_4^{(r)} = \eta_4^{(r)}\beta_5^{(r)} \tag{23}$$

$$\beta_5^{(r)}\eta_5^{(r)} = \eta_5^{(r)}\beta_4^{(r)} \tag{24}$$

$$\beta_4^{(r)}\eta_5^{(r)} = \eta_5^{(r)}\beta_5^{(r)} \tag{25}$$

$$\beta_5^{(r)}\eta_4^{(r)} = \eta_4^{(r)}\beta_4^{(r)}. \tag{26}$$

Relations (16)–(26) are characteristic of the Galilean DKP algebra. This is a new set of algebraic relations concerning DKP algebras and will be the basis for the computations in the next sections.

4. Algebraic aspects of the non-relativistic DKP equation

In this section we follow [13] in order to be able to use similar techniques to construct the phase-space picture of the Galilean covariant DKP equation. To our knowledge the present construction relating the non-relativistic representation of DKP algebra has never appeared in the literature. The above set of algebraic relations provides a new direction for studying the representations of the Galilean DKP algebras.

The action of a $\beta_\mu^{(r)}$ given in equation (16) on a $\Psi \in C_P(W)$ projects it down to the space of antisymmetric tensors of order r and $r+1$. These Ψ, which are the elements of the representation space for $C(W)$, have the general form

$$\Psi = \psi_{\mu_1 \ldots \mu_r} P^{\mu_1 \ldots \mu_r} \oplus \psi_{\mu_1 \ldots \mu_{r+1}} P^{\mu_1 \ldots \mu_{r+1}}$$

that is, each Ψ is a direct sum of covariant antisymmetric terms of order r and $r+1$. Here we consider the five-dimensional space G and fix $r = 0$ to describe the scalar representation, such that

$$\Psi = \psi P \oplus \psi_\mu P^\mu$$

namely, Ψ expresses a direct sum of scalars and vectors and can be written as

$$\Psi = \begin{pmatrix} \psi \\ \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \end{pmatrix} \tag{27}$$

where ψ is a scalar C^∞ function and ψ_μ $(\mu = 1, \ldots, 5)$ are vector functions. Note that the six-dimensional representation follows naturally from the construction. If we consider $\psi = \psi(q)$, $\psi_\mu = \partial_\mu \psi(q)$, $q \in G$, the Galilean DKP algebra (see equation (17)) has a 6×6 matrix representation given by

$$\beta^1 = \begin{pmatrix} 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 1&0&0&0&0&0 \end{pmatrix} \qquad \beta^2 = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&1&0&0&0&0 \end{pmatrix}$$

$$\beta^3 = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&1&0&0&0 \end{pmatrix} \qquad \beta^4 = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&-1&0 \end{pmatrix} \tag{28}$$

$$\beta^5 = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&-1&0&0 \end{pmatrix}$$

where we use the notation $\beta^\mu_{(r=0)} = \beta^\mu$ and from now on the value $r = 0$ in the DKP generators is understood. Furthermore, using the Galilei-invariant $\partial_\mu \beta^\mu$, and the definition of the adjoint of Ψ, i.e. $\bar{\Psi} = \Psi^* \eta_5$, where η_5 is given in equation (21), we can construct the following (real) Galilei-invariant Lagrangian,

$$\mathcal{L}_{\mathrm{DKP}} = \tfrac{1}{2}\bar{\Psi}\beta^\mu \partial_\mu \Psi - \tfrac{1}{2}(\partial_\mu \bar{\Psi})\beta^\mu \Psi + k\bar{\Psi}\Psi.$$

This Lagrangian gives rise to the equation

$$(\beta^\mu \partial_\mu + k)\Psi = 0 \tag{29}$$

and its adjoint

$$(\beta^\mu \partial_\mu + k)\bar{\Psi} = 0.$$

In the next section we obtain the phase-space representation for the Galilean-covariant DKP approach. With this purpose we rewrite equation (29), with $\Psi \in C_P(W)$ (see equation (11)) and proceed the computations in parallel with the methods used in [13] in the relativistic case. We then first need to consider

$$(\overset{\leftarrow\mu}{\beta}\partial_\mu + k)\Psi = 0 \tag{30}$$

$$(\overset{\rightarrow\mu}{\beta}\partial_\mu + k)\Psi = 0 \tag{31}$$

where $\overset{\rightarrow}{\beta}_\nu = \beta_\nu \otimes 1$ and $\overset{\leftarrow}{\beta}_\nu = 1 \otimes \beta_\nu$, with the definition $(M \otimes N)\Psi = M\Psi N$. The generators $\overset{\rightarrow}{\beta}_\nu$ and $\overset{\leftarrow}{\beta}_\nu$ satisfy the fundamental relation given in equation (17). We also define

$$\overset{+}{\beta}_\mu = \Pi_0 e_\mu + g_{\mu\nu} e^\nu \Pi_0$$

$$\overset{-}{\beta}_\mu = \Pi_0 e_\mu - g_{\mu\nu} e^\nu \Pi_0$$

which satisfy the two algebras

$$\overset{+}{\beta}_\mu \overset{+}{\beta}_\nu \overset{+}{\beta}_\rho + \overset{+}{\beta}_\rho \overset{+}{\beta}_\nu \overset{+}{\beta}_\mu = g_{\mu\nu} \overset{+}{\beta}_\rho + g_{\rho\nu} \overset{+}{\beta}_\mu \tag{32}$$

$$\overset{-}{\beta}_\mu \overset{-}{\beta}_\nu \overset{-}{\beta}_\rho + \overset{-}{\beta}_\rho \overset{-}{\beta}_\nu \overset{-}{\beta}_\mu = -g_{\mu\nu} \overset{-}{\beta}_\rho - g_{\rho\nu} \overset{-}{\beta}_\mu \, . \tag{33}$$

Hence, in order to relate $\overset{+}{\beta}$ and $\overset{-}{\beta}$ to $\overset{\rightarrow}{\beta}$ and $\overset{\leftarrow}{\beta}$, which satisfy the fundamental relation (17), we introduce from

$$\eta = \eta_1\eta_2\eta_3\eta_4\eta_5$$

the elements

$$\overset{\rightarrow}{\eta} = \overset{\rightarrow}{\eta}_1 \overset{\rightarrow}{\eta}_2 \overset{\rightarrow}{\eta}_3 \overset{\rightarrow}{\eta}_4 \overset{\rightarrow}{\eta}_5 \qquad \overset{\leftarrow}{\eta} = \overset{\leftarrow}{\eta}_1 \overset{\leftarrow}{\eta}_2 \overset{\leftarrow}{\eta}_3 \overset{\leftarrow}{\eta}_4 \overset{\leftarrow}{\eta}_5$$

with $[\overset{\rightarrow}{\eta}, \overset{\leftarrow}{\eta}] = 0$, $\overset{\rightarrow}{\eta}_\nu = \eta_\nu \otimes 1$ and $\overset{\leftarrow}{\eta}_\nu = 1 \otimes \eta_\nu$. It follows that $w = \overset{\rightarrow}{\eta} \overset{\leftarrow}{\eta}$ is such that $w^2 = 1$ and we can identify, according to equations (17), (32) and (33)

$$\overset{+}{\beta}_\mu = \overset{\rightarrow}{\beta}_\mu \qquad \overset{-}{\beta}_\mu = w \, \overset{\leftarrow}{\beta}_\mu. \tag{34}$$

Now, observe that with $\overset{+}{\beta}_\mu$ and $\overset{-}{\beta}_\mu$ we can define

$$b^+_\mu = \tfrac{1}{2}(\overset{+}{\beta}_\mu - \overset{-}{\beta}_\mu) \qquad b_\mu = \tfrac{1}{2}(\overset{+}{\beta}_\mu + \overset{-}{\beta}_\mu) \tag{35}$$

where we have adopted the conventions

$$\Pi_0 e_\mu = b_\mu \qquad g_{\mu\nu} e^\nu \Pi_0 = b^+_\mu.$$

These operators b_μ and b^+_μ fulfil the anticommutation algebra

$$\left[b^+_\mu, b_\nu\right]_+ = (P) g_{\mu\nu} + b^+_\mu b_\nu \tag{36}$$

$$\left[b^+_\mu, b^+_\nu\right]_+ = [b_\mu, b_\nu]_+ = 0. \tag{37}$$

Note that, if we consider a matrix representation, the projector $\Pi_0 \equiv P$ onto scalars is identified to the unity $\mathbf{1}$ given by the matrix with entries, $P_{rs} = 1$, for $r = s = 1$; $P_{rs} = 0$ otherwise.

The construction of the operators b_μ and b^+_μ allows us to follow directly the techniques introduced in [16] to arrive at the phase-space picture of spinning-particle equations. Furthermore, they appear as a new way of representing the DKP algebras. Their relations with β are analogous to the ones that creation and annihilation operators hold with γ generators in the Clifford algebra.

5. DKP equation in the generalized phase space

As mentioned earlier, a number of papers [4, 13, 16–18] have been focused on obtaining relativistic Liouville–von Neumann equations with the purpose of investigating the effect of internal degrees of freedom in a phase-space context. In this section we continue to use those similar techniques in order to construct a non-relativistic DKP version of those phase-space theories. In order to pass from equations (30) and (31) to the Liouville–von Neumann equation in the phase-space variables (q, p), we shall use the Wigner–Moyal transform. We take as the domain of the DKP operators the space of functions of the form of equation (27) where the coefficient functions ψ and ψ_μ are defined in the configuration space G. The way we combine the algebraic calculus developed in last section with the usual (q, p) phase-space variables is as follows. Let (q^μ, p_μ) be the usual coordinates of a point of T^*G. We look for the set of all functions defined via the Wigner–Moyal transform

$$f(x, p) = \frac{1}{2\pi} \int g(q', q)\, \mathrm{e}^{-\mathrm{i}p_\nu \xi^\mu}\, \mathrm{d}\mu(\xi) \tag{38}$$

where

$$x^\mu = \tfrac{1}{2}(q'^\mu + q^\mu) \tag{39}$$

$$\xi^\mu = q^\mu - q'^\mu \qquad \mu = 1, \ldots, 5 \tag{40}$$

and $\mathrm{d}\mu(\xi)$ is a measure in ξ-space.

Now the generalized phase space can be defined as a product space

$$T := T^*G \otimes D_r^{r+1}(W)$$

where $D_r^{r+1}(W)$ has been defined in section 3. In the case, in question, $r = 0$ and $r + 1 = 1$. Note that T is the total space of states of a particle with continuous degrees of freedom represented by the (x, p) variables and also internal discrete degrees of freedom represented by the algebra $D_0^1(W)$. A typical function on this generalized phase space has the form

$$F = F(x, p)P + F_\mu(x, p)P^\mu$$

where the $F's$ are given by the Wigner–Moyal transform as defined above. The set of all these functions constitute a common domain for the operators like $\left(x, p, \partial/\partial x, \partial/\partial p, b_\mu^+, b_\mu\right)$. Note that the functions g in equation (38) are now being seen as the components of the density matrix $\rho = \Psi \otimes \bar{\Psi}$, with Ψ given by equation (27).

Next we use the mathematical structure developed in sections 3 and 4 in order to derive the DKP equations in the (x, p) variables. We go back to equations (30) and (31) but with ρ instead Ψ, and write

$$\frac{\partial}{\partial x'^\mu} \overset{\rightarrow\mu}{\beta} \rho(x', x) + k\rho(x', x) = 0 \tag{41}$$

$$\frac{\partial}{\partial x^\mu} \overset{\leftarrow\mu}{\beta} \rho(x', x) + k\rho(x', x) = 0 \tag{42}$$

where the β are already specialized for the case $r = 0$. Multiplying equation (42) by w and using definitions in equation (34), we obtain

$$\frac{\partial}{\partial x'^\mu} \overset{+}{\beta}{}^\mu \rho(x', x) + k\rho(x', x) = 0$$

$$\frac{\partial}{\partial x^\mu} \overset{-}{\beta}{}^\mu \rho(x', x) + wk\rho(x', x) = 0.$$

Next, using the relations given in equation (35), we have

$$(b^\mu + b^{\mu+})\left[\frac{\partial}{\partial x'^\mu} + k\right]\rho(x', x) = 0$$

$$(b^\mu - b^{\mu+})\left[\frac{\partial}{\partial x^\mu} + kw\right]\rho(x', x) = 0.$$

Performing the change of variables given in equations (39) and (40), it results

$$(b^\mu + b^{\mu+})\left(\frac{1}{2}\frac{\partial}{\partial x^\mu} - \frac{\partial}{\partial \xi^\mu}\right)\rho(\xi, x) + k\rho(\xi, x) = 0$$

$$(b^\mu - b^{\mu+})\left(\frac{1}{2}\frac{\partial}{\partial x^\mu} - \frac{\partial}{\partial \xi^\mu}\right)\rho(\xi, x) + kw\rho(\xi, x) = 0.$$

The addition and subtraction of these equations lead to

$$\left[\left(b^\mu \frac{\partial}{\partial x^\mu} - 2b^{\mu+}\frac{\partial}{\partial \xi^\mu}\right) + k(1-w)\right]\rho(\xi, x) = 0$$

$$\left[\left(b^{+\mu} \frac{\partial}{\partial x^\mu} - 2b^{\mu}\frac{\partial}{\partial \xi^\mu}\right) + k(1+w)\right]\rho(\xi, x) = 0$$

which, under the Wigner–Moyal transformation, equation (38), become

$$\left[\left(b^\mu \frac{\partial}{\partial x^\mu} - 2ib^{\mu+}p_\mu\right) + k(1-w)\right]F(x, p) = 0 \tag{43}$$

$$\left[\left(b^{+\mu} \frac{\partial}{\partial x^\mu} - 2ib^{\mu}p_\mu\right) + k(1+w)\right]F(x, p) = 0. \tag{44}$$

Since $w^2 = 1$, equation (44) can be written as

$$L^{[\mu]}F(x, p) = kF(x, p) \tag{45}$$

where

$$L^{[\mu]} = \left(-\frac{b^{+\mu}}{2}\frac{\partial}{\partial x^\mu} + ib^\mu p_\mu\right). \tag{46}$$

The use of the upper dumb index μ in $L^{[\mu]}$ used in between $[\cdot]$ is just to indicate how we are going to compute the square of the operator, i.e. equation (47) below. Equations (43) and (44) are DKP equations in the generalized phase space. To derive the corresponding Liouville operator, we compute the square of $L^{[\mu]}$ given by

$$L^2 = L^{[\mu]} \circ L^{[\nu]} = \tfrac{1}{2}(L^{[\mu]}L^{[\nu]} + L^{[\nu]}L^{[\mu]}). \tag{47}$$

Noting that $w^2 = 1$, we have, using equation (45) and (36)

$$L_{\mathrm{DKP}}F(x, p) = k^2 F(x, p) \tag{48}$$

where

$$L_{\mathrm{DKP}} = L^2 = [b^{+\mu}, b^\nu]_+ p_\nu \frac{\partial}{\partial x^\mu} = ((P)g^{\mu\nu} + b^{+\mu}b^\nu)p_\nu\frac{\partial}{\partial x^\mu} \tag{49}$$

$$:= L^s_{\mathrm{DKP}} + L^v_{\mathrm{DKP}} \tag{50}$$

with the notations $L^s_{\mathrm{DKP}} = (P)g^{\mu\nu}\frac{\partial}{\partial x^\mu}$ and $L^v_{\mathrm{DKP}} = b^{+\mu}b^\nu p_\nu\frac{\partial}{\partial x^\mu}$. Equation (48) is the Wigner phase-space representation of the non-relativistic DKP equation, L_{DKP} is the Liouville operator and $F(x, p)$ is the corresponding (DKP-spinor) Wigner function. Taking a matrix representation we have a direct sum of the scalar part and the vector one, i.e. L_{DKP} is decomposed into two blocks. Thus for the scalar part we have

$$(P)g^{\mu\nu}p_\nu \frac{\partial}{\partial x^\mu} F(x, p) = (P)p_\nu \frac{\partial}{\partial x^\nu} F(x, p) = 0.$$

Then writing $F(x, p) = f(x, p)_w$, and taking into account that the projector P is just the unity on its range, we have

$$p_\nu \frac{\partial}{\partial x^\nu} f(x, p)_w \equiv \{p^2, f(x, p)_w\}_M = 0$$

where $\{\cdot, \cdot\}_M$ is the Moyal bracket given by

$$\{g(x, p), h(x, p)\}_M = g(x, p) * h(x, p) - h(x, p) * g(x, p)$$

with

$$g(x, p) * h(x, p) = g(x, p) \exp\left[\mathrm{i} \left(\frac{\overleftarrow{\partial}}{\partial q} \frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p} \frac{\overrightarrow{\partial}}{\partial q} \right) \right] h(x, p).$$

Considering the embedding of $\mathbf{R}^3$ in (4 + 1)-de Sitter space (see [23]),

$$x^\mu = \left(\mathbf{x}, vt, \frac{\mathbf{x}^2}{2vt} \right) \tag{51}$$

$$p^\mu = \left(\mathbf{p}, mv, \frac{H}{v} \right) \tag{52}$$

$$p_\mu = \left(\mathbf{p}, -\frac{H}{v}, -mv \right) \tag{53}$$

$$f_w(x, p) = f_w(\mathbf{x}, \mathbf{p}, t) \tag{54}$$

we find the usual equation for the evolution of the Wigner function, $f_w(\mathbf{x}, \mathbf{p}, t)$, describing a free particle,

$$\frac{\partial}{\partial t} f_w(\mathbf{x}, \mathbf{p}, t) + \frac{1}{m} \mathbf{p} \cdot \nabla f_w(\mathbf{x}, \mathbf{p}, t) = 0. \tag{55}$$

The remaining block which corresponds to

$$b^{+\mu} b^\nu p_\nu \frac{\partial}{\partial x^\mu} \tag{56}$$

accounts for the vector part. It operates on the remaining components of $F(q, p)$. Therefore, equation (55) is obtained from the algebraic form of the Liouville operator given by equation (49) which shows the consistency of our developments. In this case the change in the evolution of the (DKP-spinor) Wigner function $F(x, p)$ comes exclusively from the multiplicative p_ν and differential operators $\frac{\partial}{\partial x^\mu}$.

Now, it is well known that the notion of spinning particle is closely connected with the notion of interaction. The spinning electron, for example, is only recognized as having spin when it is put into interaction with an electromagnetic field. It is then worth considering what happens when we perform the constructions above but for a DKP particle in the presence of an external electromagnetic field. This is investigated in the next section.

6. Minimal coupling gauge field

We take into account a gauge field by considering the minimal coupling $-\mathrm{i}\partial_\mu \rightarrow -\mathrm{i}\partial_\mu - eA_\mu$ (c = 1) into equations (41) and (42). We also use the approximation $A_\mu(q) = A_\mu(x) + \xi^\nu/2\partial A_\mu(x)/\partial x^\nu$ which is proper to describe harmonic interaction. Hence, we obtain for the Liouville–von Neumann operator

$$\begin{aligned} L_{\mathrm{DKP}} = {} & [b^{+\nu}, b^\mu]_+(p_\mu - eA_\mu)\frac{\partial}{\partial x^\nu} - \frac{e}{2}(b^{+\nu}b^\mu - b^{+\mu}b^\nu)\left(\frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}\right) \\ & + e[b^{+\nu}, b^\mu]_+(p_\mu - eA_\mu)\frac{\partial A_\nu}{\partial x^\alpha}\frac{\partial}{\partial p_\alpha} = ((P)g^{\mu\nu} + b^{+\nu}b^\mu)(p_\mu - eA_\mu)\frac{\partial}{\partial x^\nu} \\ & - \frac{e}{2}(b^{+\nu}b^\mu - b^{+\mu}b^\nu)\left(\frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}\right) \\ & + e((P)g^{\mu\nu} + b^{+\nu}b^\mu)(p_\mu - eA_\mu)\frac{\partial A_\nu}{\partial x^\alpha}\frac{\partial}{\partial p_\alpha} \end{aligned} \tag{57}$$

where we have followed a similar procedure as the one developed in the last section. Expression (57) can also be split into the form $L^s_{\mathrm{DKP}} + L^v_{\mathrm{DKP}}$ with

$$L^s_{\mathrm{DKP}} = (P)g^{\mu\nu}(p_\mu - eA_\mu)\frac{\partial}{\partial x^\nu} + e(P)g^{\mu\nu}(p_\mu - eA_\mu)\frac{\partial A_\nu}{\partial x^\alpha}\frac{\partial}{\partial p_\alpha}$$

and

$$\begin{aligned} L^v_{\mathrm{DKP}} = {} & b^{+\nu}b^\mu(p_\mu - eA_\mu)\frac{\partial}{\partial x^\nu} + eb^{+\nu}b^\mu(p_\mu - eA_\mu)\frac{\partial A_\nu}{\partial x^\alpha}\frac{\partial}{\partial p_\alpha} \\ & - \frac{e}{2}(b^{+\nu}b^\mu - b^{+\mu}b^\nu)\left(\frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}\right). \end{aligned}$$

It can be seen that the operator L_{DKP} obtained above has a complete resemblance with the relativistic one [13] except for the fact that here all the algebraic generators refer to the DKP algebra built from the five-dimensional space G associated with the metric (1). In particular, we note that the term $(b^{+\nu}b^\mu - b^{+\mu}b^\nu)$ in L^v_{DKP} corresponds to a 'spin' operator in analogy to the relativistic case [13]. The total Liouville operator is $L_{\mathrm{DKP}} = L^s_{\mathrm{DKP}} + L^v_{\mathrm{DKP}}$, which gives rise to the Liouville–von Neumann equation for $F(x, p)$ in the presence of an external electromagnetic field, i.e.

$$L_{\mathrm{DKP}}F(x, p) = 0. \tag{58}$$

Note that despite we are analysing the scalar case ($r = 0$), there is in equation (58) the vector part ($r = 1$). This is a characteristic of the relativistic DKP algebra which also holds here in the Galilean DKP algebra. As a consequence, we see that $F(x, p)$ changes not only due to the changes in position and momentum in a trajectory, but also because of the presence of the term

$$\frac{e}{2}(b^{+\nu}b^\mu - b^{+\mu}b^\nu)\left(\frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}\right)$$

containing the electromagnetic field $\mathcal{F}_{\mu\nu}$. This term generates what is equivalent to a rotation (Galilean) transformation among the components of (DKP-spinor) Wigner function. In other words, a solution $F(x, p)$, now considered as a field

$$F = F(x, p) + F_\mu(x, p)P^\mu$$

also undergoes a transformation originated from the action of the DKP generators on its components.

One can see here a justification for the use of the generalized phase-space approach as a way to formulate and give meaning to the notion of internal degrees of freedom in the context of the Wigner function.

7. Concluding remarks

Starting with a five-dimension manifold G endowed with the metric (1), we used the notion of geometric algebras to derive the Galilean DKP phase-space representation and its associated Wigner function. In order to accomplish this program, we first deduced new algebraic relations for the DKP algebras in the non-relativistic realm. We then used these as follows. Initially we analyse the formalism for free particles and later we extend it for a particle interacting with an external electromagnetic field, via a minimal coupling. In both cases the Liouville operator and the equation of motion for the Wigner function are written explicitly. In particular, the result for the free particle shows that if we use the embedding of $\mathbf{R}^3$ in the five-dimensional space G, we obtain as expected the usual equation for the evolution of the Wigner function $f_w(\mathbf{x}, \mathbf{p}, t)$. This shows the consistence of our algebraic developments. Of particular importance is the Liouville operator (57) of the Liouville–von Neumann equation (58) which shows explicitly how internal degrees of freedom associated with Galilean DKP particles can be analysed in a phase-space setting. The importance of this relies on the fact that there is an extended symplectic structure on the generalized phase space that can be recovered from our expression of the Liouville operator. In other words, we can use the Liouville operators (49) and (57) to reobtain equations (3)–(6) of section 2 but now for the extended phase space. The resulting expression allows us to see how the Poisson bracket and hence the symplectic structure are extended in a way to incorporate the internal degrees of freedom. Thinking along these lines we cannot leave without mentioning the use of the method of coadjoint orbits as another way to give a meaning to the notion of internal degrees of freedom associated with symplectic structures. In this approach the coadjoint orbits of the symmetry groups are shown to be symplectic spaces [32–34] and the classification of such orbits provides a theory of elementary particles with internal degrees of freedom. The structure of these orbits and their connection with the derivation of relativistic wave equations can be seen in [33]. A study aiming to write equations (3)–(6) from our Liouville operator and further comparison with the method of coadjoint orbits is now in progress and will be published elsewhere.

Concluding, we could mention that from a general stand point the algebraic characterization of phase space, using yet another approach [35, 36], has been of interest in situations as for instance when one looks for developing the kinetic theory [37] and to describe a quantum field theory in a curved spacetime background[38, 39] or to treat a non-Abelian gauge field theory at a thermal non-equilibrium regime [40, 41]. In this respect the algebraic procedure, we have been adopting, offers an interesting possibility to be further investigated in a forthcoming paper.

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